Model of pulsar pair cascades in **non uniform electric fields**: growth rate, density profile and screening time

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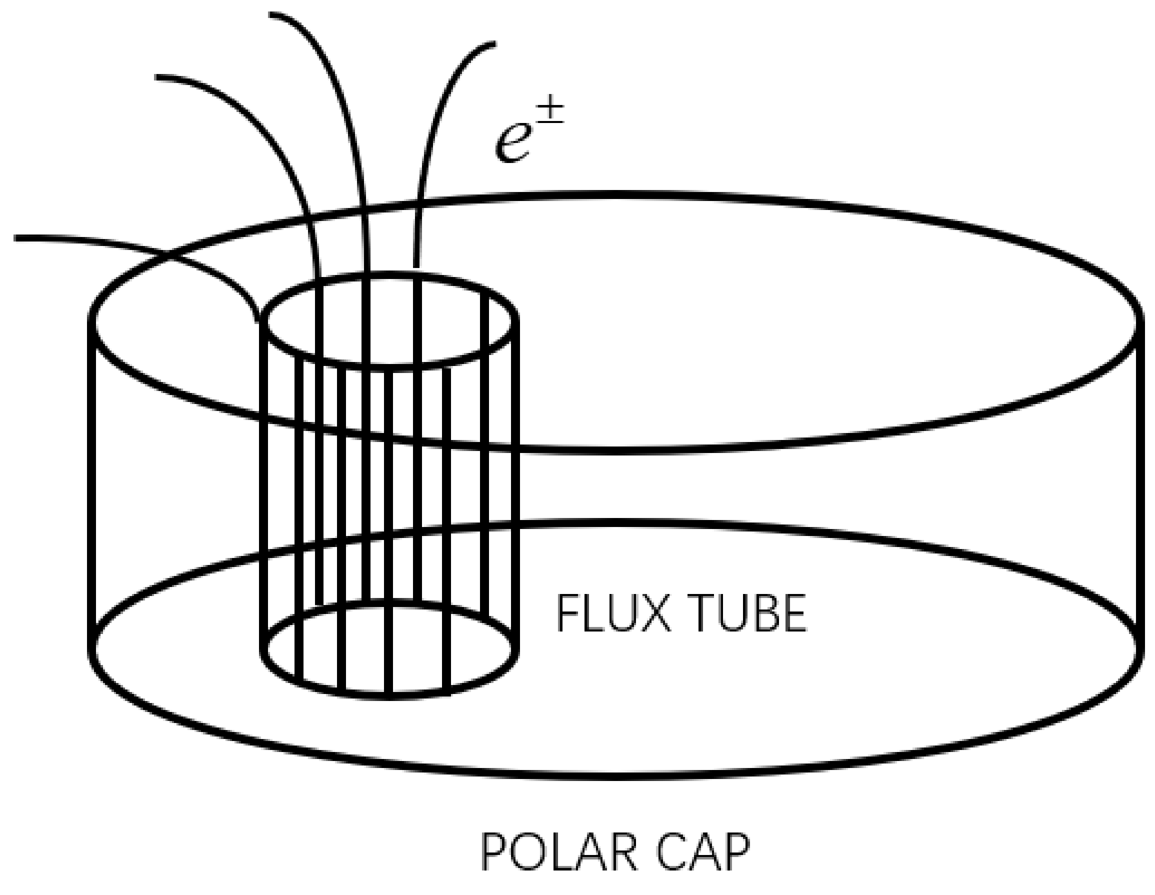
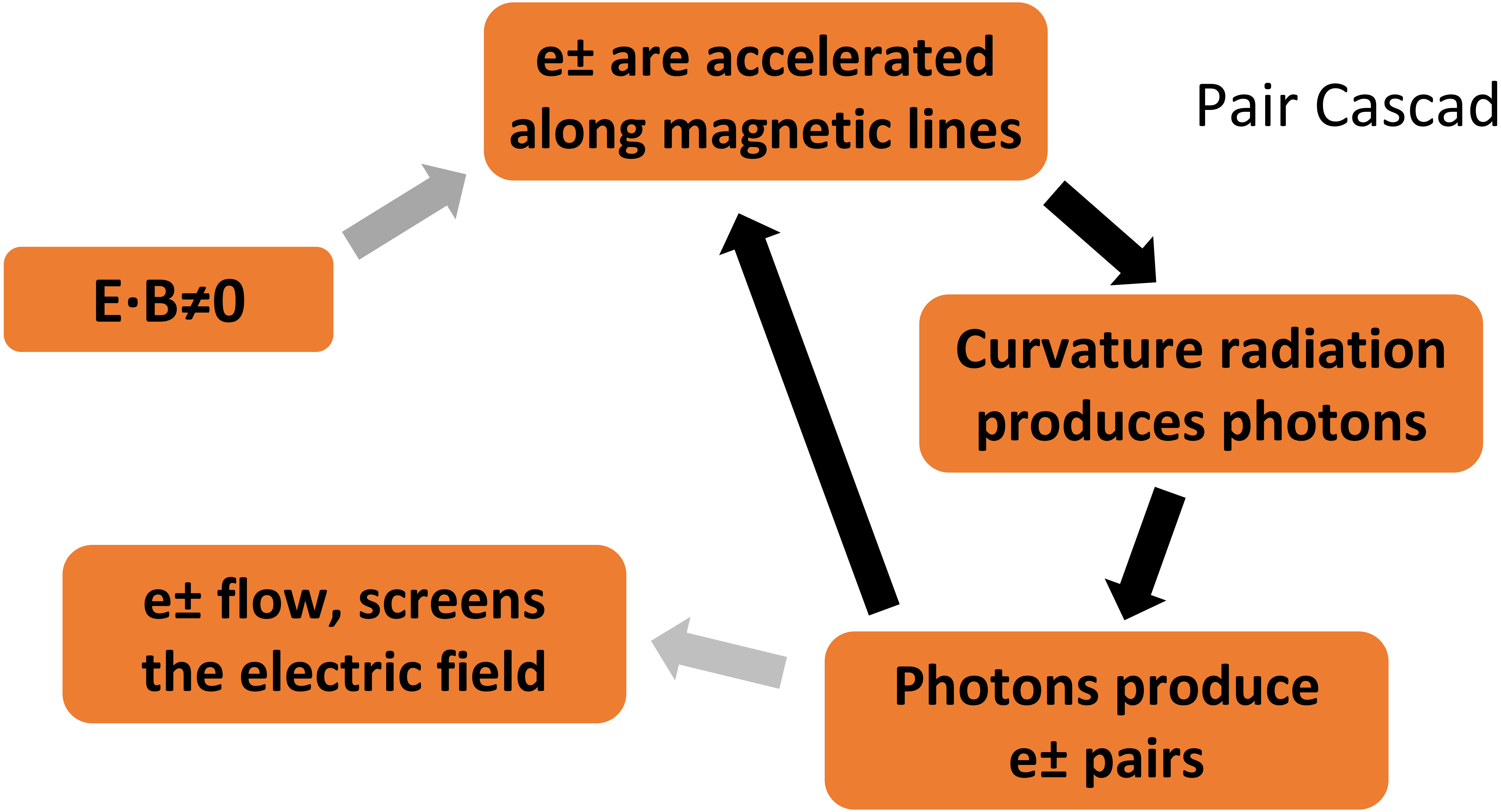
I. Introduction:

Pulsars have strong magnetic field, but magnetized plasma's flow makes magnetospheres **force-free** (E·B~0) almost everywhere.

Theories (like RS75) predict that there exists places **where E·B≠0** (such as vacuum gaps), at which a series of particle processes happen.



Pair Cascade



Bunches of electrons/positrons flow out of vacuum gap, produce coherent radio radiation……

Difficulties in computer simulating pulsar pair cascades:
——large multiplicity
——vacuum gaps (~100m) much larger than shortest plasma kinetic scales (~1cm)

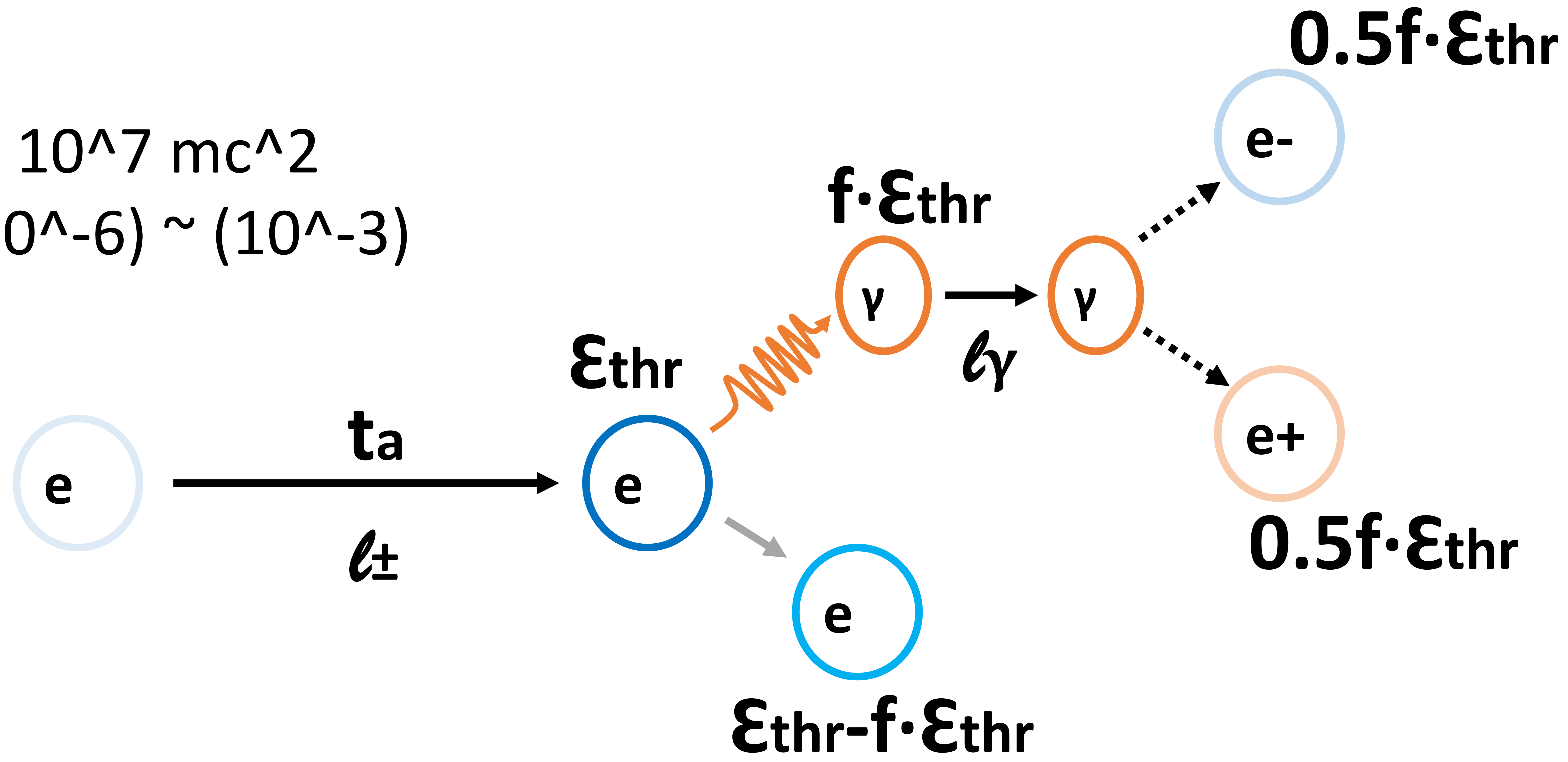
One solution is using **heuristic models**, with **PIC simulation.**

The authors suppose a **threshold energy** for the particle processes, and begin their analytical and numerical study.

II. Cascade in a uniform electric field:

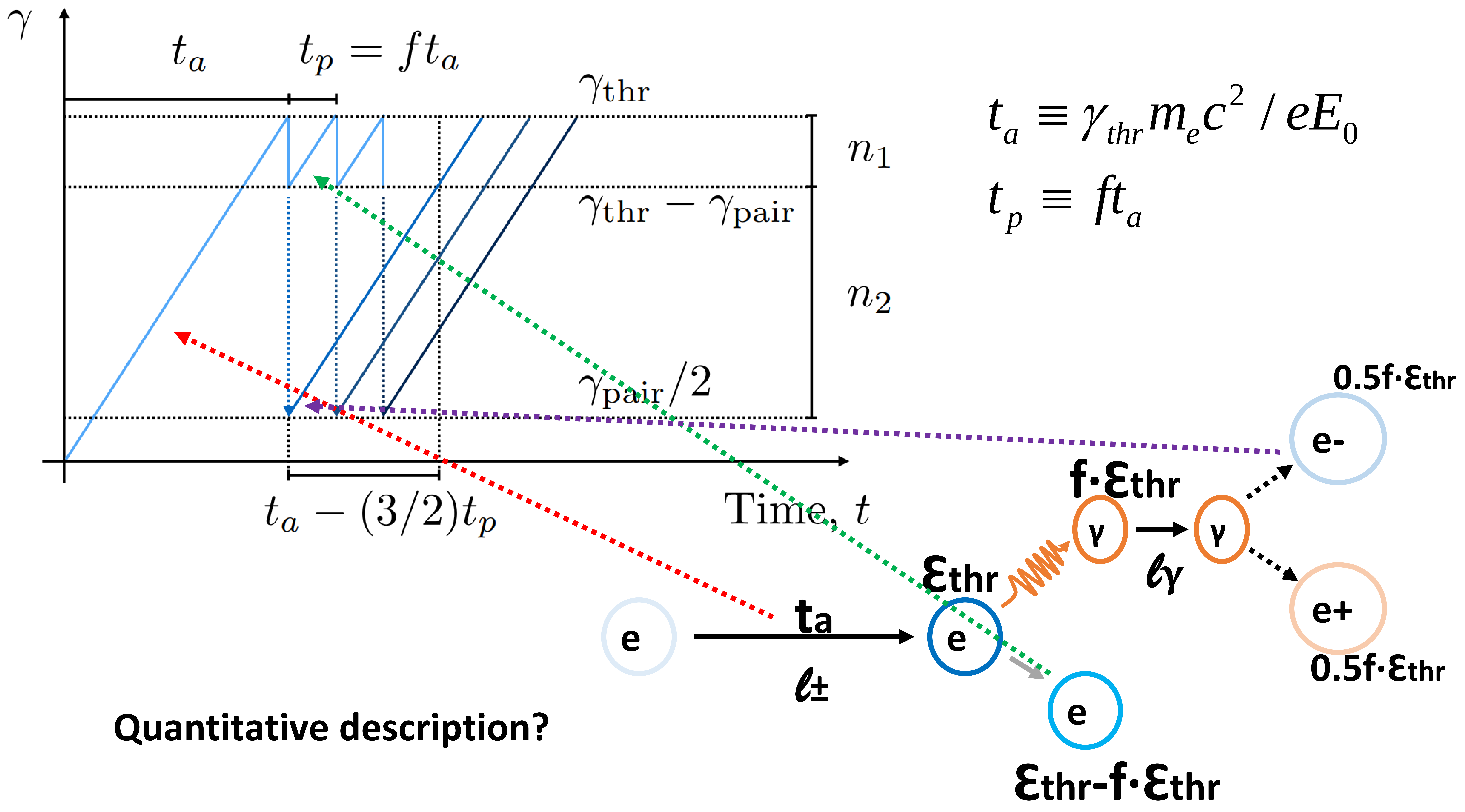
$\varepsilon_{thr} \approx 10^7\, mc^2$

$f \approx (10^{-6}) \sim (10^{-3})$



$\ell_\pm \approx 100\text{m}$

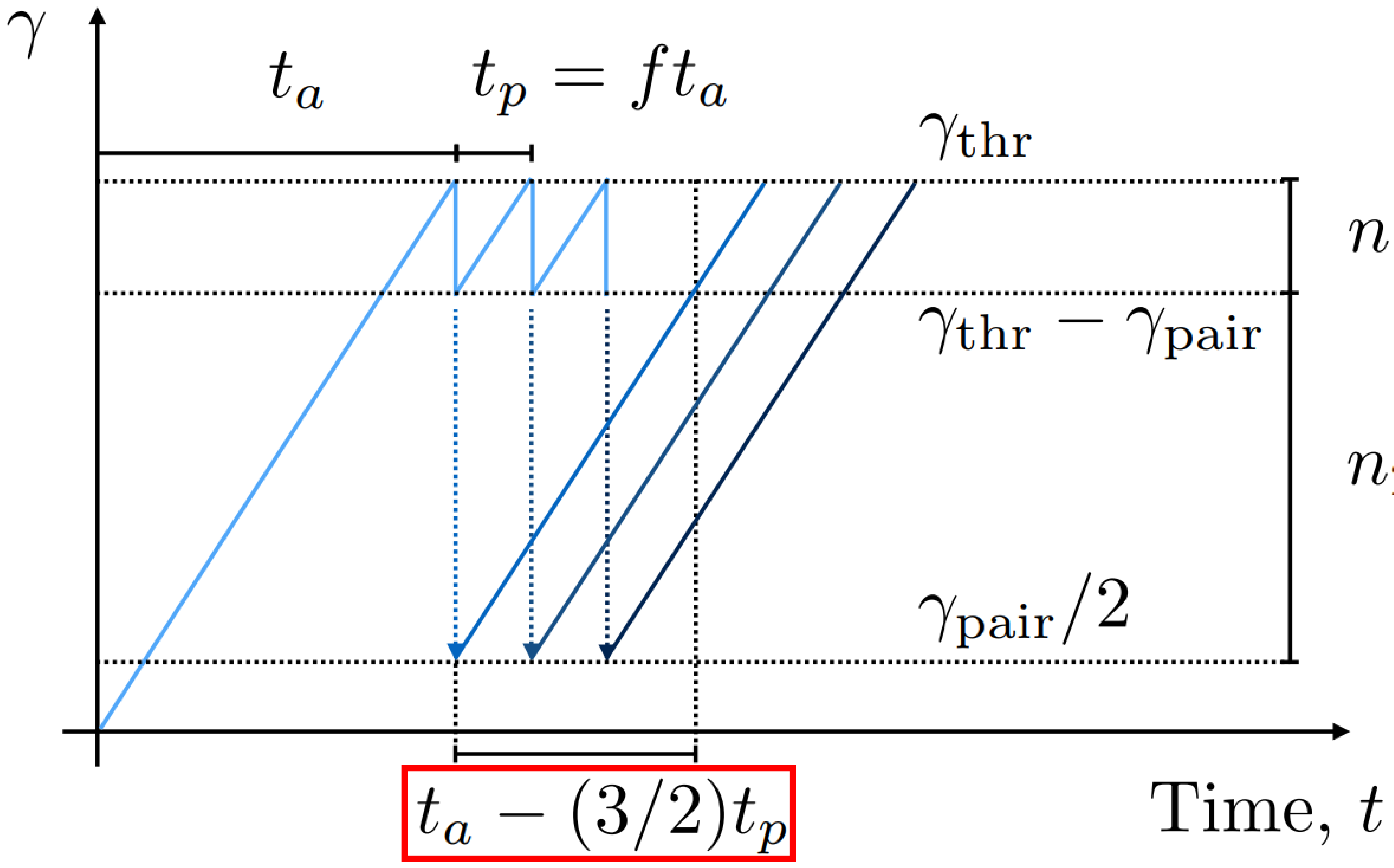
$\ell_\gamma \approx 1\sim10\text{m}$ ➔➔➔ Neglect ℓ_γ



$$t_a \equiv \gamma_{thr} m_e c^2 / eE_0$$

$$t_p \equiv f t_a$$

Quantitative description?



$n_1(t)$ $\gamma \in [\gamma_{\rm thr} - \gamma_{\rm pair}, \gamma_{\rm thr}]$

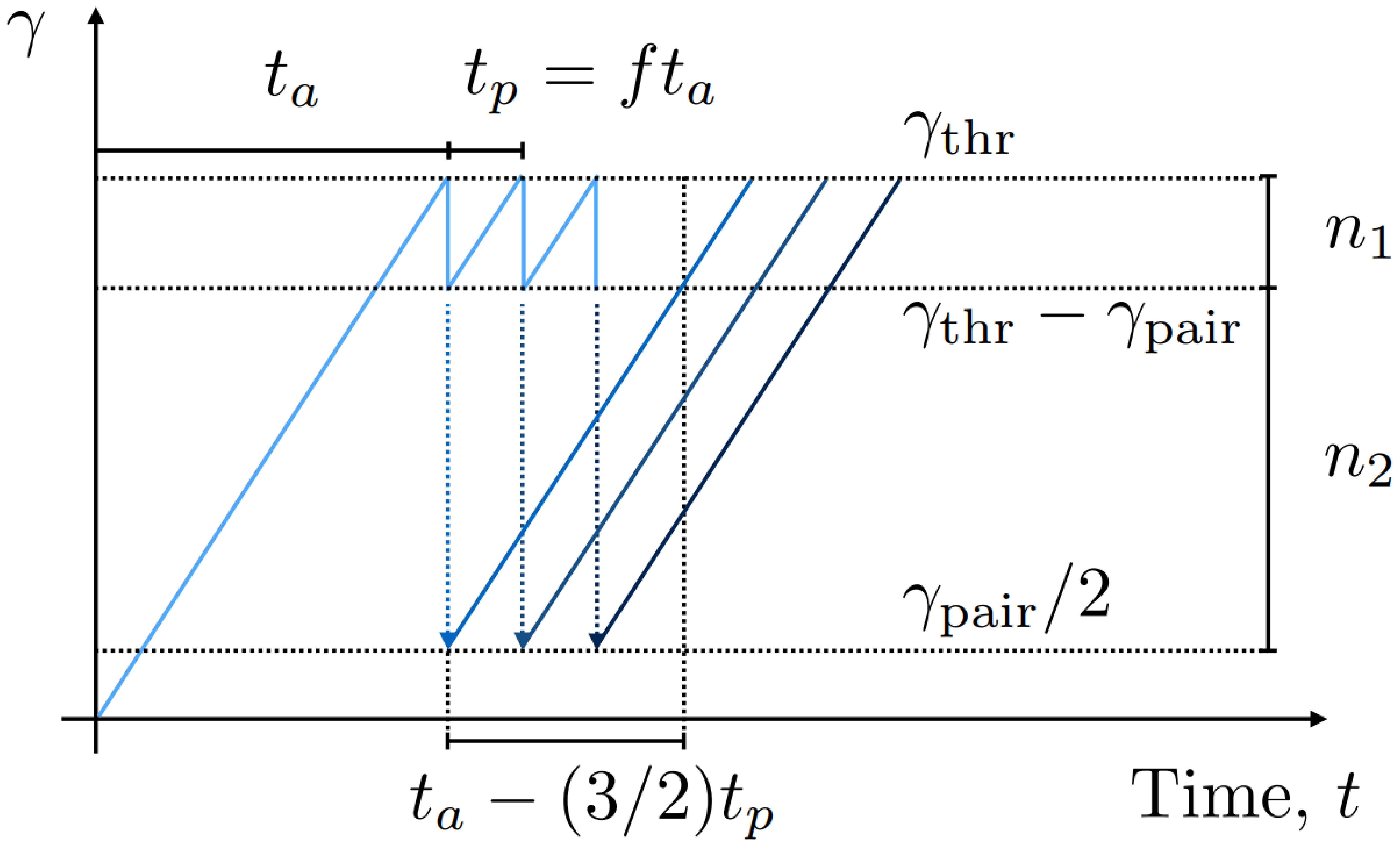
$n_2(t)$ $\gamma < \gamma_{\rm thr} - \gamma_{\rm pair}$

After $t_a(1 - 3f/2)$, all n2 particles change into n1.

$n_1 \neq> n_2$

$n_2 \Rightarrow n_1$

$$n_1(t + t_a(1 - 3f/2)) \simeq n_1(t) + n_2(t)$$



After $f t_a$, all n1 particles produce two n2 particles.

After $t_a(1 - 3f/2)$, all n2 particles change into n1.

$$\frac{\mathrm{d}n_2(t)}{\mathrm{d}t} \simeq \frac{2n_1(t)}{f t_a} - \frac{n_2(t)}{(1-3f/2)t_a}$$

$$\begin{cases} n_1(t + t_a(1 - 3f/2)) \simeq n_1(t) + n_2(t) \\ \dfrac{\mathrm{d}n_2(t)}{\mathrm{d}t} \simeq \dfrac{2n_1(t)}{ft_a} - \dfrac{n_2(t)}{(1 - 3f/2)t_a} \end{cases}$$

Try: $n_{1,2}(t) \propto \exp(\Gamma t)$

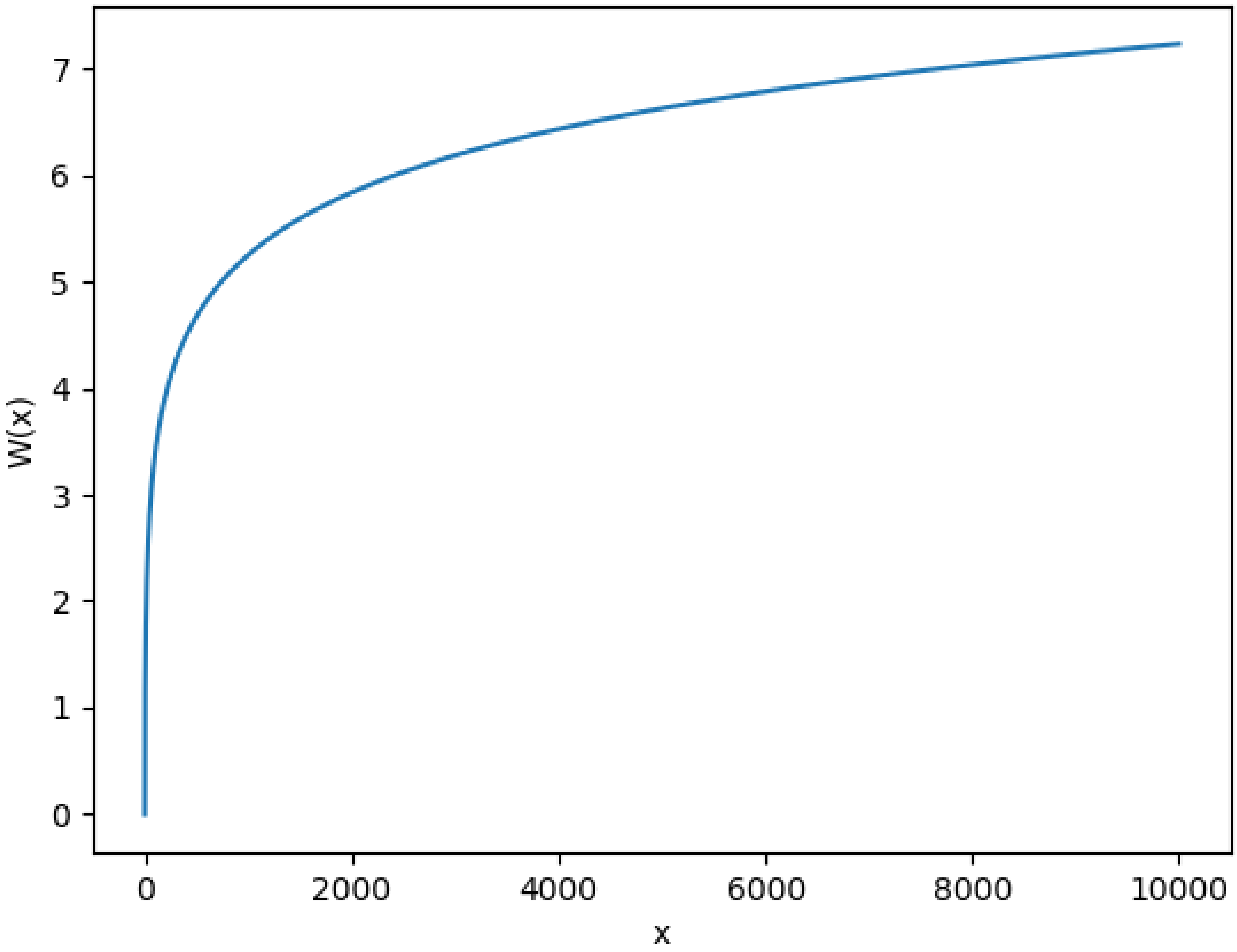
➔➔➔ $f \cdot t_a \Gamma = \dfrac{2}{e^{\Gamma t_a (1-3f/2)} - 1} - \dfrac{f}{1-3f/2}$ ➔➔➔ (f<<1) $f \Gamma t_a = \dfrac{2}{e^{\Gamma t_a}}$

We have: $\Gamma t_a \simeq W(2/f) \simeq \ln(2/f)$

W: Lambert function.

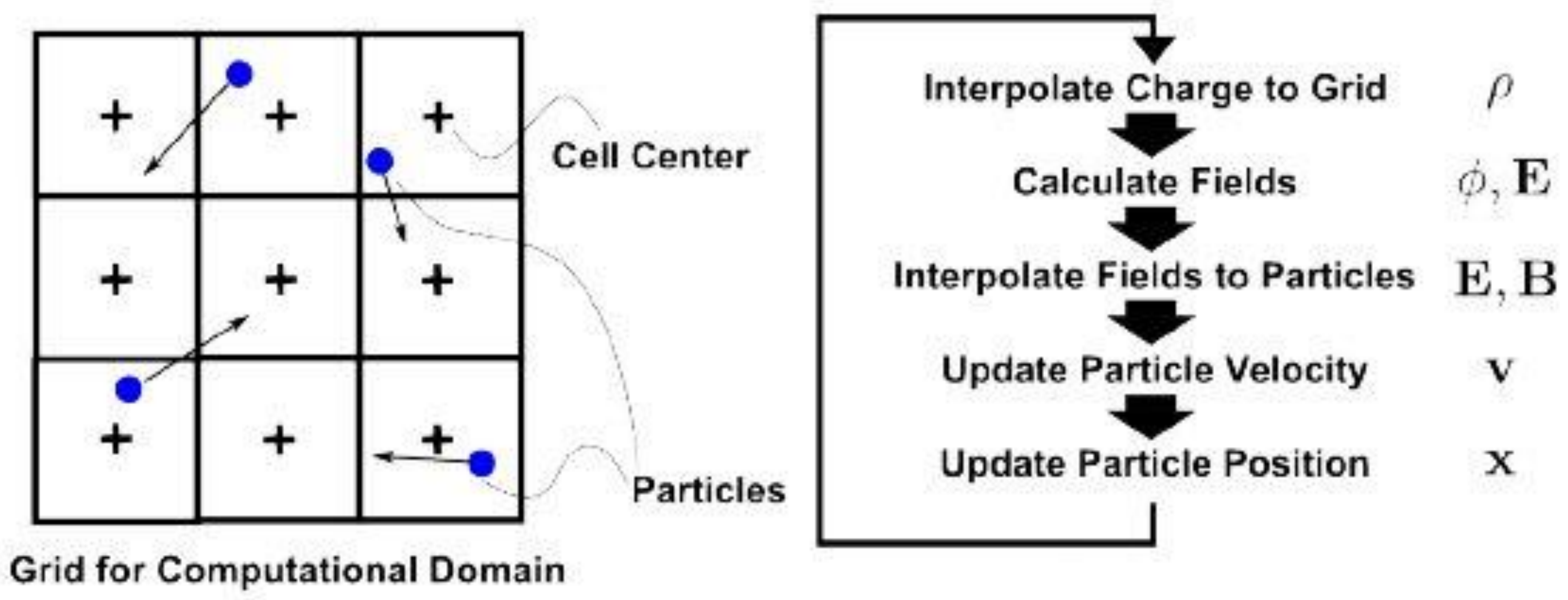
z=W(z)e^(W(z))

An exponentially growing solution.


W(x)
7
6
5
4
3
2
1
0
0
2000
4000
6000
8000
10000
x

Simulation: 1D particle-in-cell, with OSIRIS (Fonseca et al. 2002)

Particle-in-cell (PIC) method:



Ebersohn et al. 2014

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\vec{r}_{12} \quad \Rightarrow \quad \vec{F} = q\left(\vec{E} + \vec{v}\times\vec{B}\right)$$

O(N^2) O(N)

Uniform E field E_0:

$$eE_0(c/\omega_p)/m_e c^2 \simeq 3000 \gg 1$$
$$\omega_p = (4\pi e^2 n_0/m_e)$$

Simulation domain length:

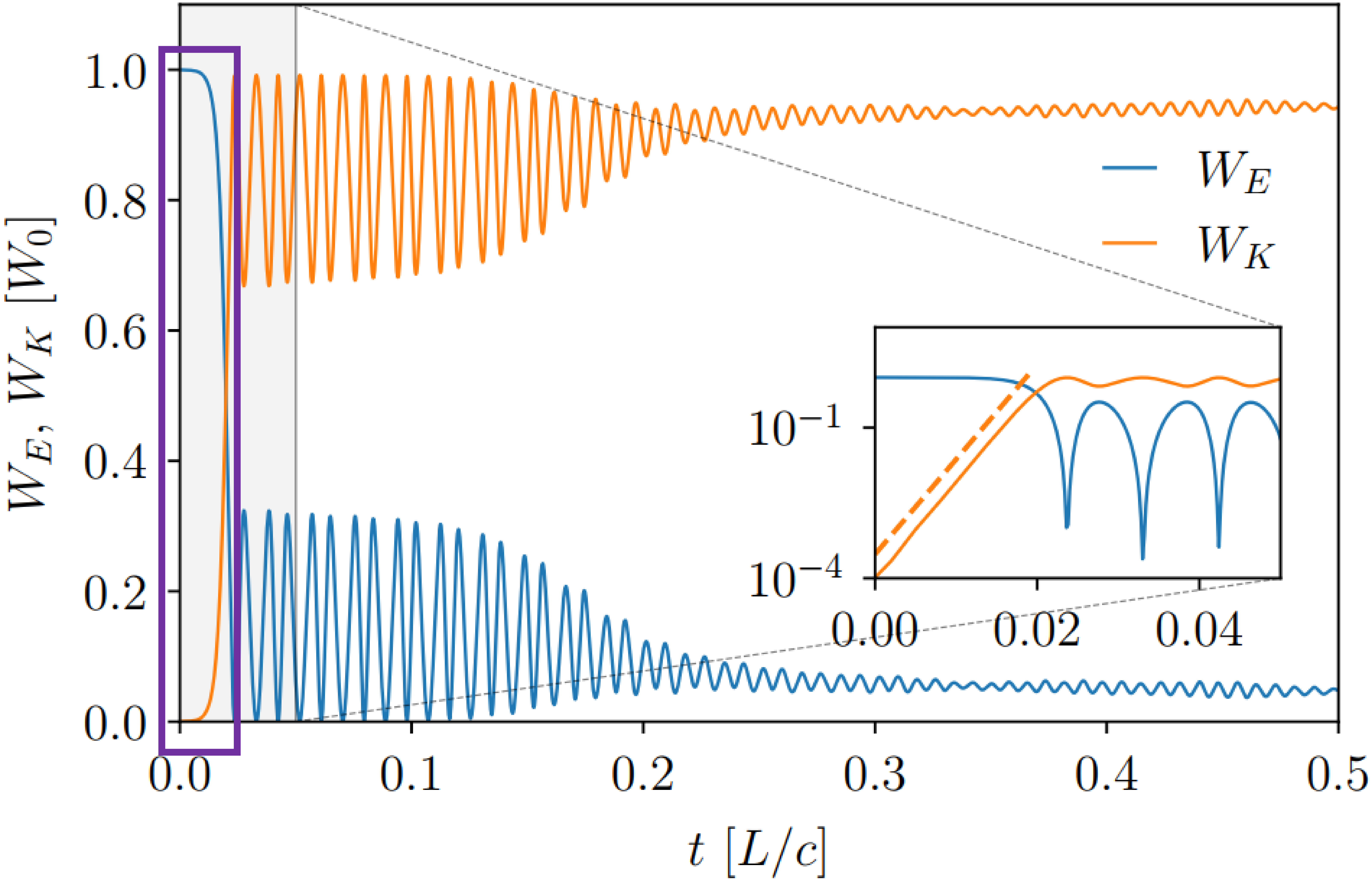
$$L/(c/\omega_p) \simeq 30$$

Grid resolution:

$$\Delta x/(c/\omega_p) = 0.015$$

$$\gamma_{\rm thr} = 500 \text{ and } f = 0.1$$

Simulation result:

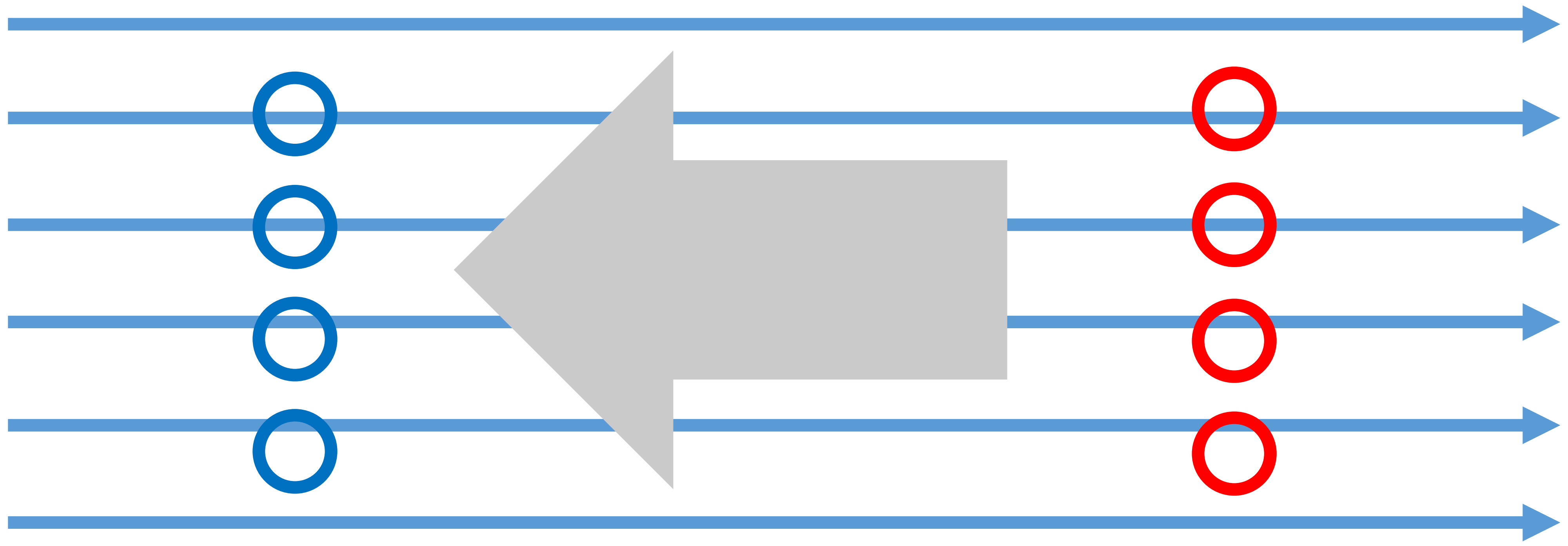


$$W_E = \int_0^L (E^2/8\pi)\ \mathrm{d}x$$

$$W_K = \sum_i (\gamma_i - 1) m_e c^2$$

After 0.02L/c ?

The e± number growing ➔➔➔current growing ➔➔➔screen E field



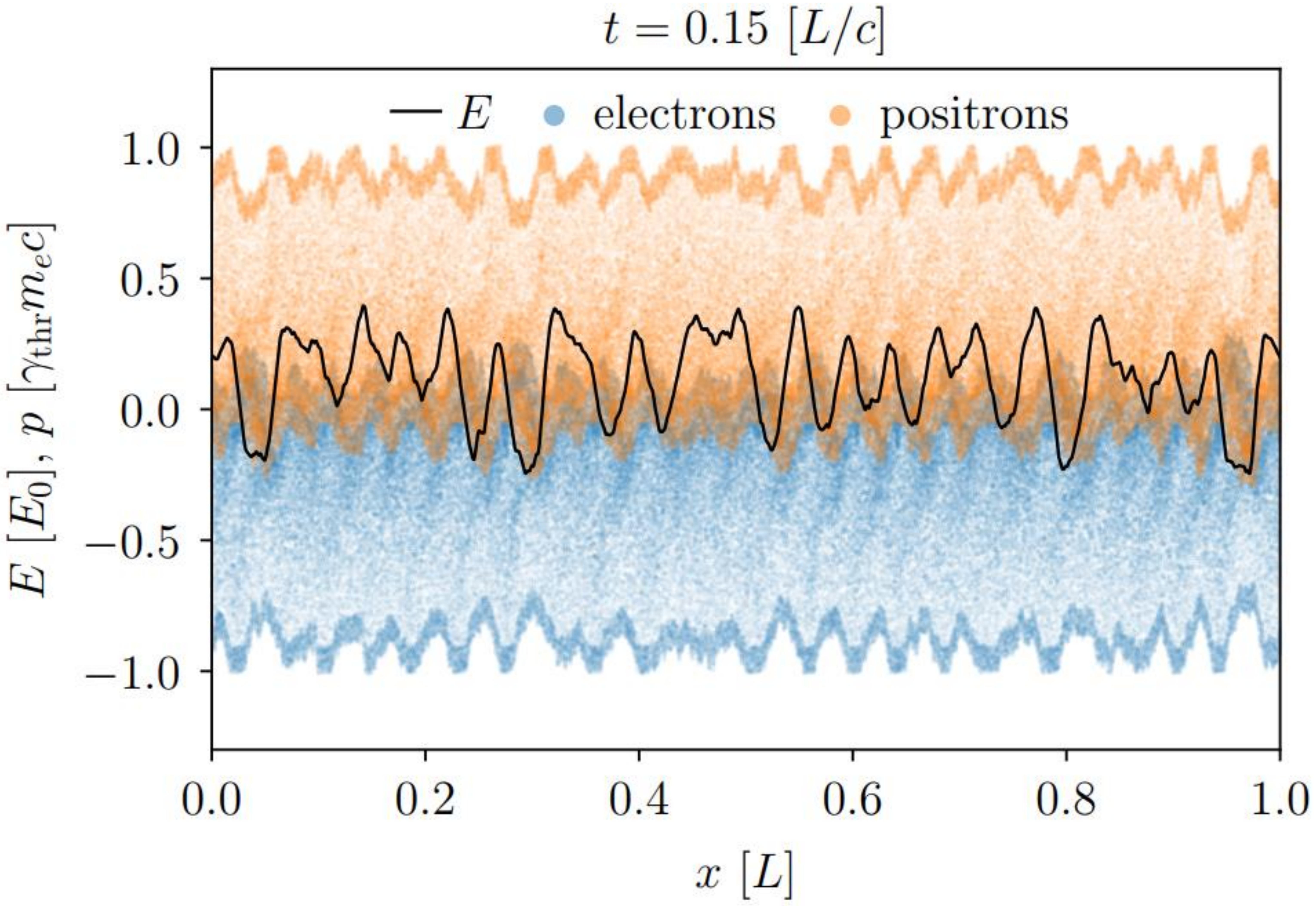
$$\frac{\partial E}{\partial t} = -4\pi j \simeq 8\pi e c n_{\pm}$$

➔➔➔The reversed E field decelerates e±, prevent growing…

E field begins **oscillating**, no new e± produced.

➔➔➔The instable perturbations accelerate some e± again, making pair production, dumping E field.

Phase space at a certain time:



Perturbations ➔ reacceleration ➔ pair production

$$\gamma_{\mathrm{thr}} = 500 - 5000$$
$$f = 10^{-3} - 0.1$$

→ Similar results.

Next step: more complex and realistic E field.

III. Cascade in a linear electric field:

Consider a 1D vacuum gap near pulsar surface.

Assume $\hat{\Omega} = \hat{\mu}$

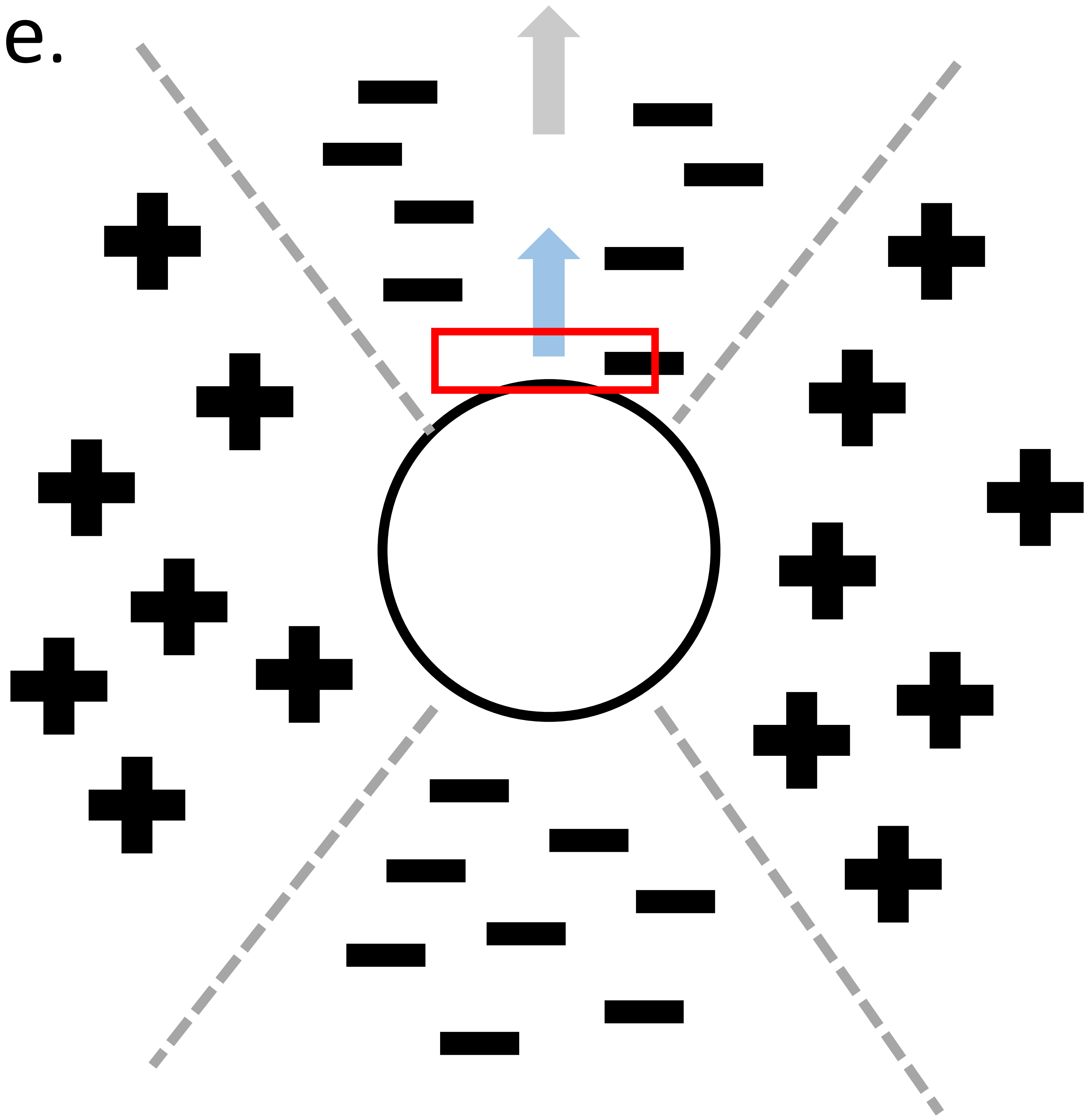
In the corotating frame:

$$\frac{\partial E}{\partial x} = 4\pi(\rho - \rho_{\mathrm{GJ}})$$

< 0

$$\frac{\partial E}{\partial t} = -4\pi(j - j_{\mathrm{m}})$$

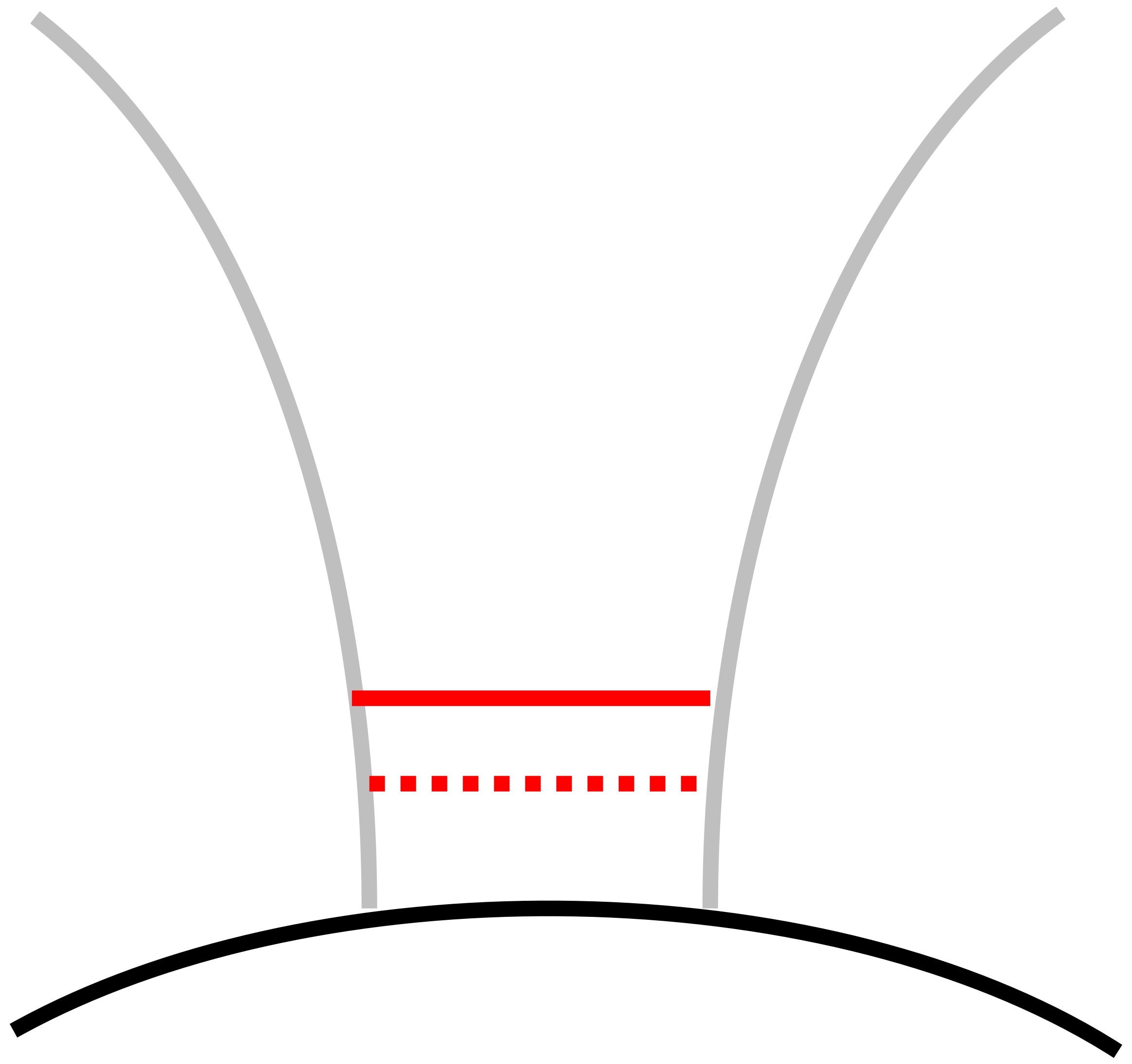
magnetosphere



Positrons $\rho_+ = r|\rho_{\rm GJ}|$ inflow, making the gap grow at a velocity v_f:

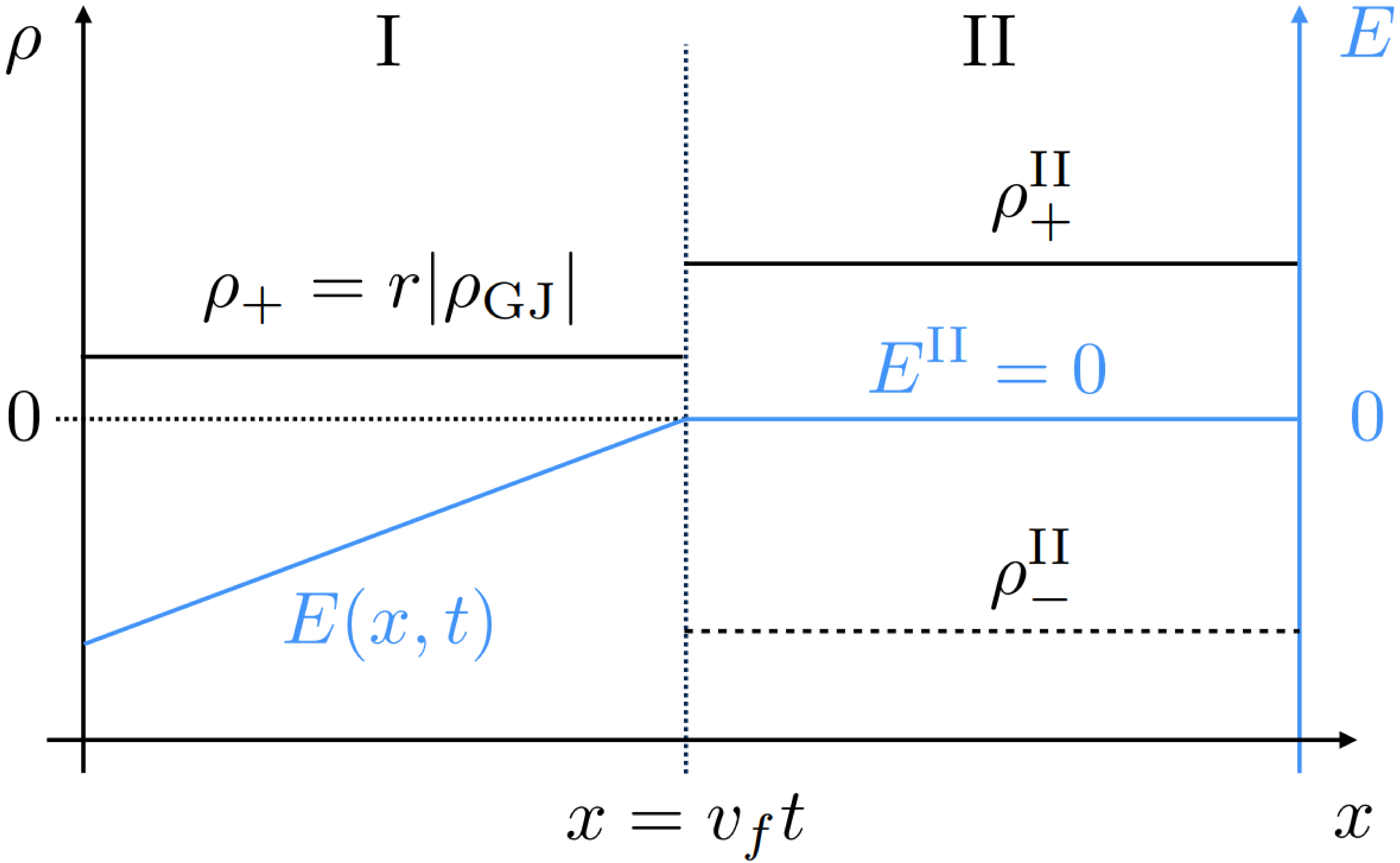
$$\frac{\partial E}{\partial x} = 4\pi(\rho - \rho_{\rm GJ}) \qquad \rho = \rho_+ = -r\rho_{GJ}$$

$$\frac{\partial E}{\partial t} = -4\pi(j - j_{\rm m}) \qquad j = \rho v_f \quad j_m = \rho_{GJ} v_f$$



$$E(x,t) = \begin{cases} 4\pi|\rho_{\rm GJ}|(1+r)(x - v_f t)\ , x < v_f t \\ 0\ , x \geq v_f t\ , \end{cases}$$

Pair cascade requires: $1/3 < v_f/c < 1$ (Beloborodov 2008) (Timokhin and Arons 2012)

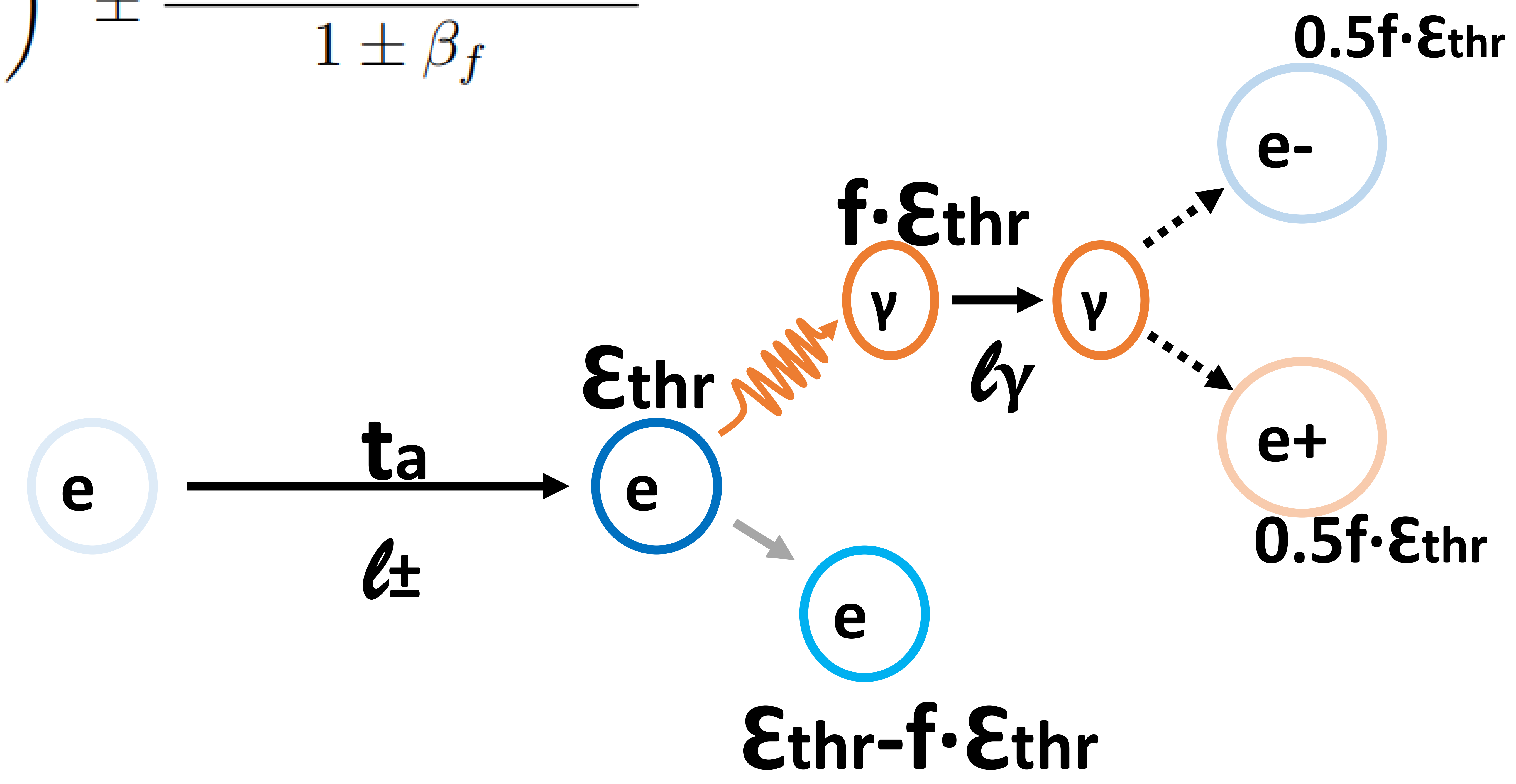

ρ
I
II
E
ρ_+^{II}
$\rho_+ = r|\rho_{\mathrm{GJ}}|$
$E^{\mathrm{II}} = 0$
0
0
ρ_-^{II}
$E(x,t)$
$x = v_f t$
x

E=E(x,t) ➔➔➔ $t_{a\pm} = t_{a\pm}(x_i, t_i)$ and $t_{p\pm} = t_{p\pm}(x_i, t_i)$

$$t_{a\pm}(x_i,t_i) \simeq \frac{t_i\beta_f - x_i/c}{1\pm\beta_f} - \sqrt{\left(\frac{t_i\beta_f - x_i/c}{1\pm\beta_f}\right)^2 \pm \frac{\gamma_{\rm thr}/(1+r)\omega_{p,\rm GJ}^2}{1\pm\beta_f}}$$

$$\beta_f \equiv v_f/c$$

Very complex…



Analytical attempt: consider a thin layer, t_a and t_p slowly evolve.

$v_f/c \gtrsim 0.7$ and $f \lesssim 0.05$

We have:

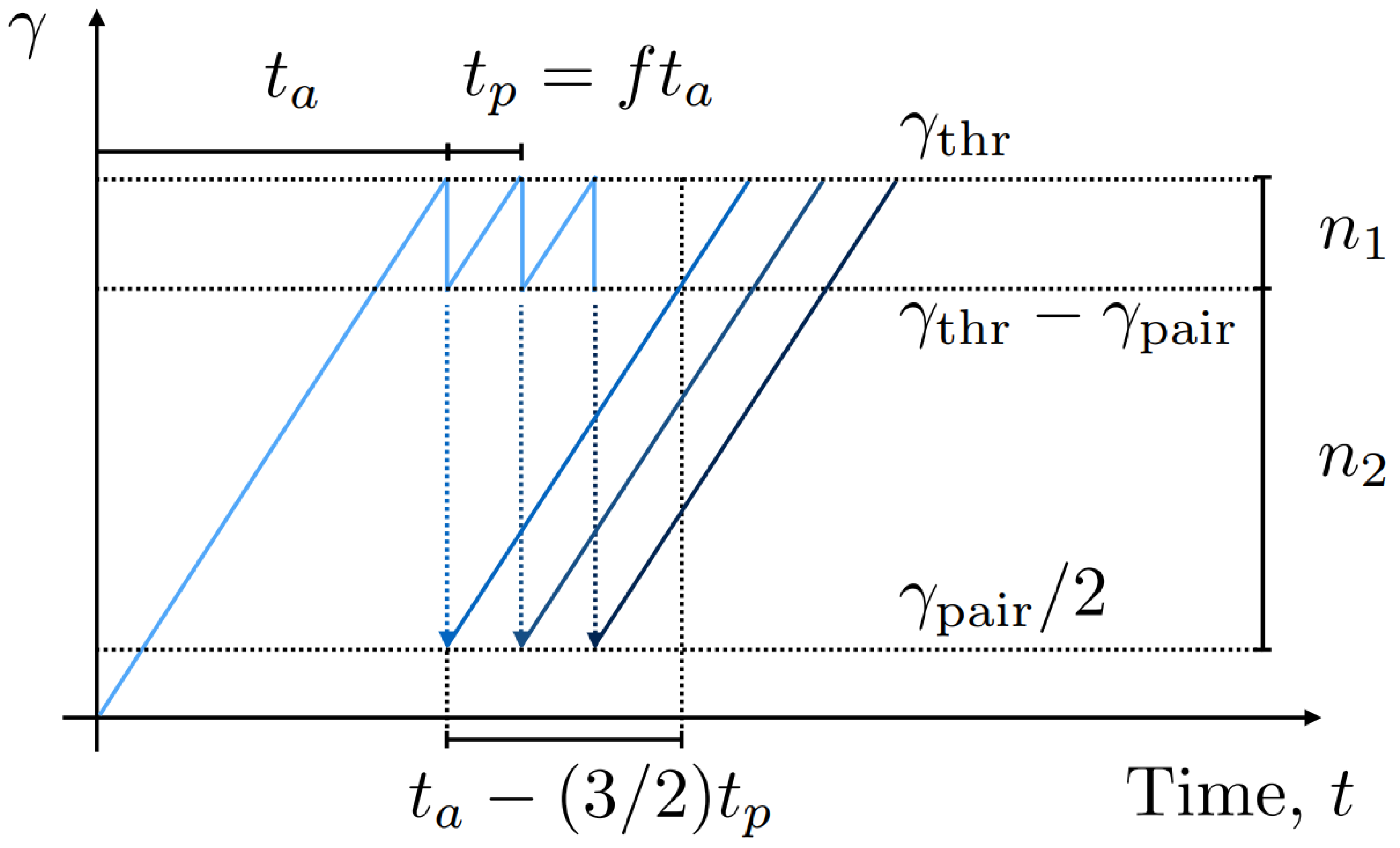
$$t_p(t) \simeq f t_a(t)$$

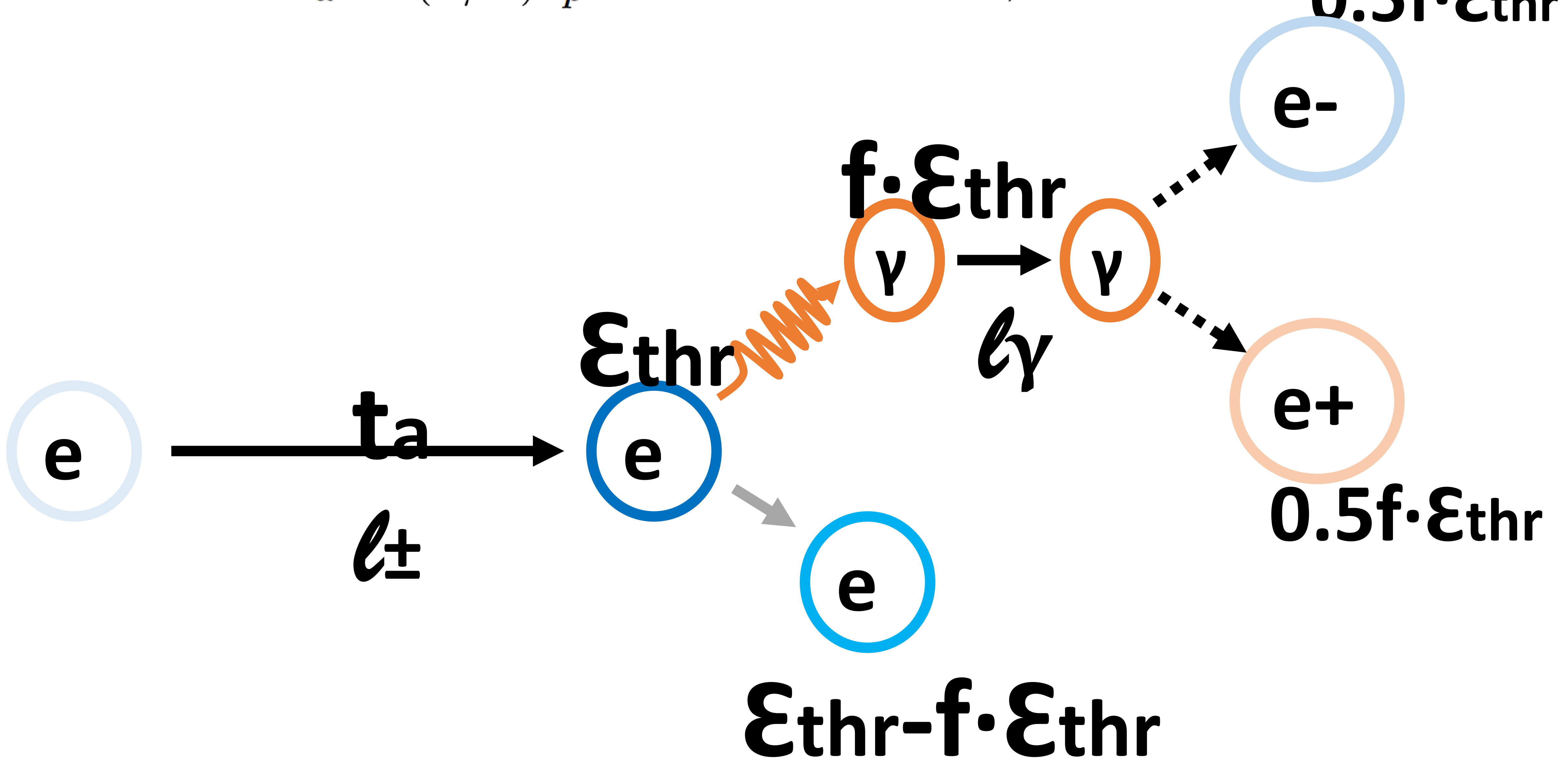
$$t_a(t) - 3t_p(t)/2 \simeq t_a(t)$$

Then:

$$\begin{cases} n_1(t + t_a(1-3f/2)) \simeq n_1(t) + n_2(t) \\ \frac{\mathrm{d}n_2(t)}{\mathrm{d}t} \simeq \frac{2n_1(t)}{f t_a} - \frac{n_2(t)}{(1-3f/2)t_a} \end{cases}$$

$$\begin{cases} n_1(t + t_a(t)) \simeq n_1(t) + n_2(t) \ , \\ \frac{\mathrm{d}n_2(t)}{\mathrm{d}t} \simeq \frac{n_1(t)}{f t_a(t)} - \frac{n_2(t)}{t_a(t)} \ . \end{cases}$$





Plug in (WKB approximation):

$$n_{1,2}(t) \propto \exp\left(\int_{t_a^*}^{t} \Gamma(t')\ \mathrm{d}t'\right)$$

t_a^* : the time n(t) start exponentially growing.

Then we have:

$$\frac{n_2(t)}{n_1(t)} = \frac{1}{f(\Gamma(t)t_a(t) + 1)}$$

Assume Γ varies slowly during t_a: $\Gamma(t') = \Gamma(t) + (t' - t)\dot{\Gamma}(t)$

$$\int_t^{t+t_a(t)} \Gamma(t')\,\mathrm{d}t' \simeq \Gamma(t)t_a(t)\left(1 + \frac{\dot{\Gamma}(t)t_a(t)}{2\Gamma(t)}\right)$$

$$\equiv \Gamma(t)t_a(t)\,(1 + \psi(t)) \doteq \tilde{\Gamma}\, t_a(t)$$

$$n_1(t+t_a(t)) \simeq n_1(t) + n_2(t) \quad \Rightarrow \quad \frac{n_1(t+t_a(t))}{n_1(t)} \simeq 1 + \frac{n_2(t)}{n_1(t)}$$

Plug in: $$\frac{n_2(t)}{n_1(t)} = \frac{1}{f(\Gamma(t)t_a(t)+1)}$$

$$\Rightarrow \quad \frac{n_1(t+t_a(t))}{n_1(t)} \simeq 1 + \frac{1}{f\cdot(\Gamma(t)t_a(t)+1)} \approx \frac{1}{f\cdot\Gamma(t)t_a(t)} = \frac{1+\psi(t)}{f\cdot\tilde{\Gamma}(t)t_a(t)}$$

Then with: $n_{1,2}(t) \propto \exp\left(\int_{t_a^*}^{t} \Gamma(t')\, \mathrm{d}t'\right)$ and $\int_t^{t+t_a(t)} \Gamma(t')dt' \simeq \tilde{\Gamma}(t)t_a(t)$

$$\Rightarrow \quad \exp\left[\tilde{\Gamma}(t)t_a(t)\right] \simeq \frac{1+\psi(t)}{f\tilde{\Gamma}(t)t_a(t)}$$

Solution: $$\Gamma(t)t_a(t) = \frac{1}{1+\psi(t)} W\left(\frac{1+\psi(t)}{f}\right)$$

However, $$\psi(t) = \frac{\dot{\Gamma}(t)t_a(t)}{2\Gamma(t)}$$

Notice that t_a and t_a^* have similar meaning:

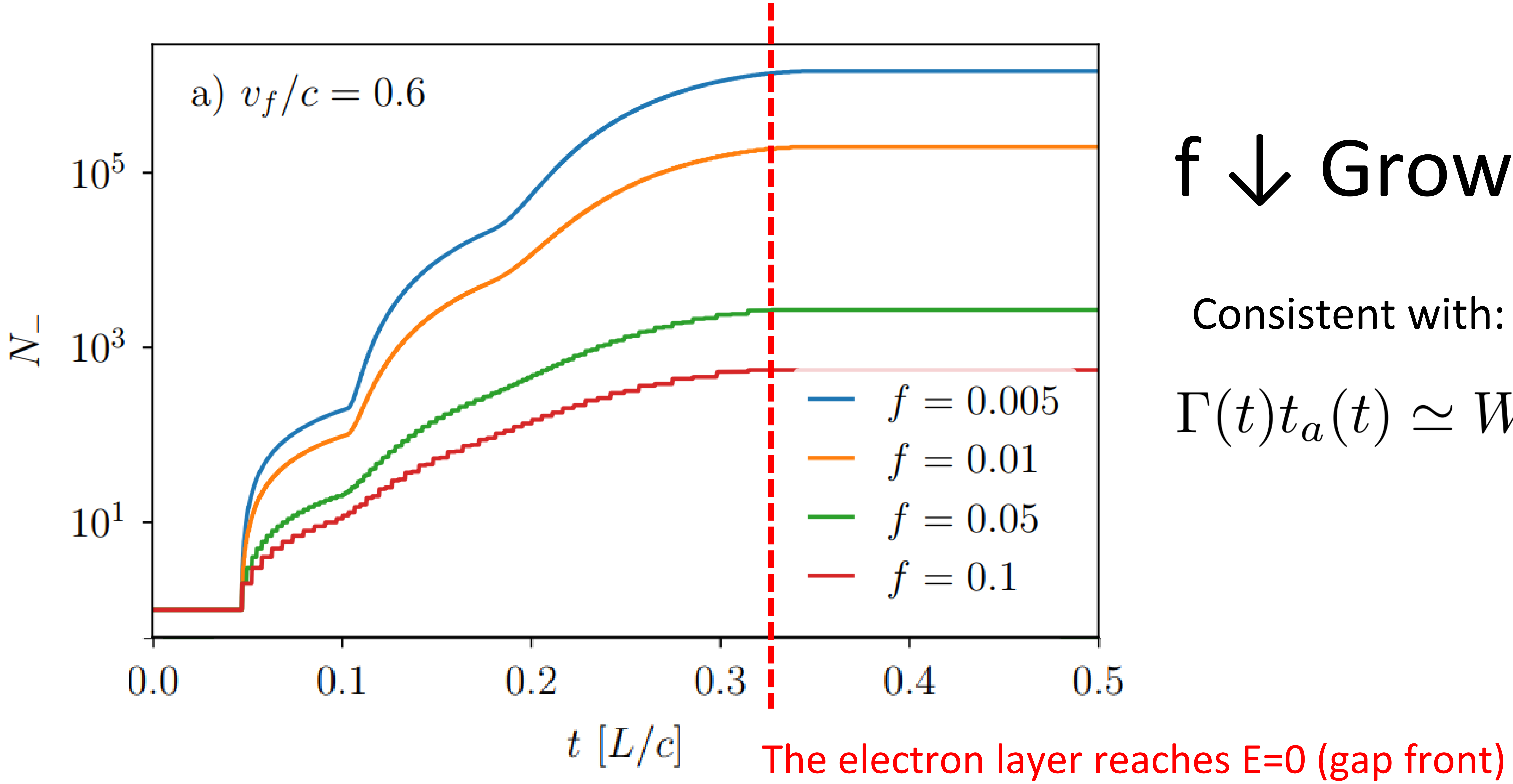
➔➔➔ Assume $t_a \approx t_a^*(1+Ct)$, $C \ll 1$

$$\psi(t) = -\frac{Ct_a^*}{2}$$

$$\Gamma(t)t_a(t) \simeq W(1/f)$$

$$\psi \simeq \frac{Ct_a^*}{W(1/f)} \ll 1$$?

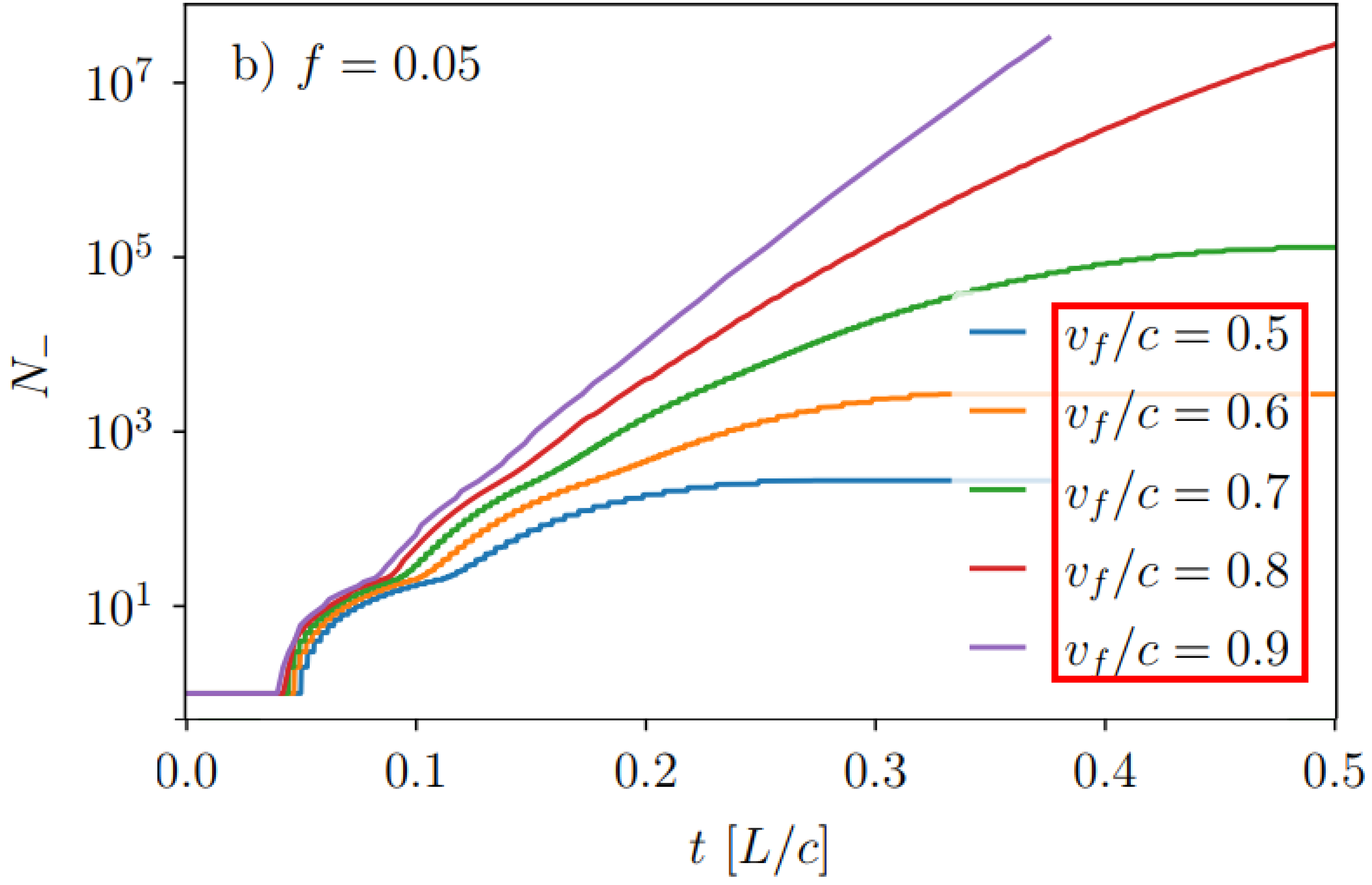
Simulation (1) —— a single electron: γ_{thr}=1000, grid solution Δx/L=0.001



f ↓ Growing ↑

Consistent with:

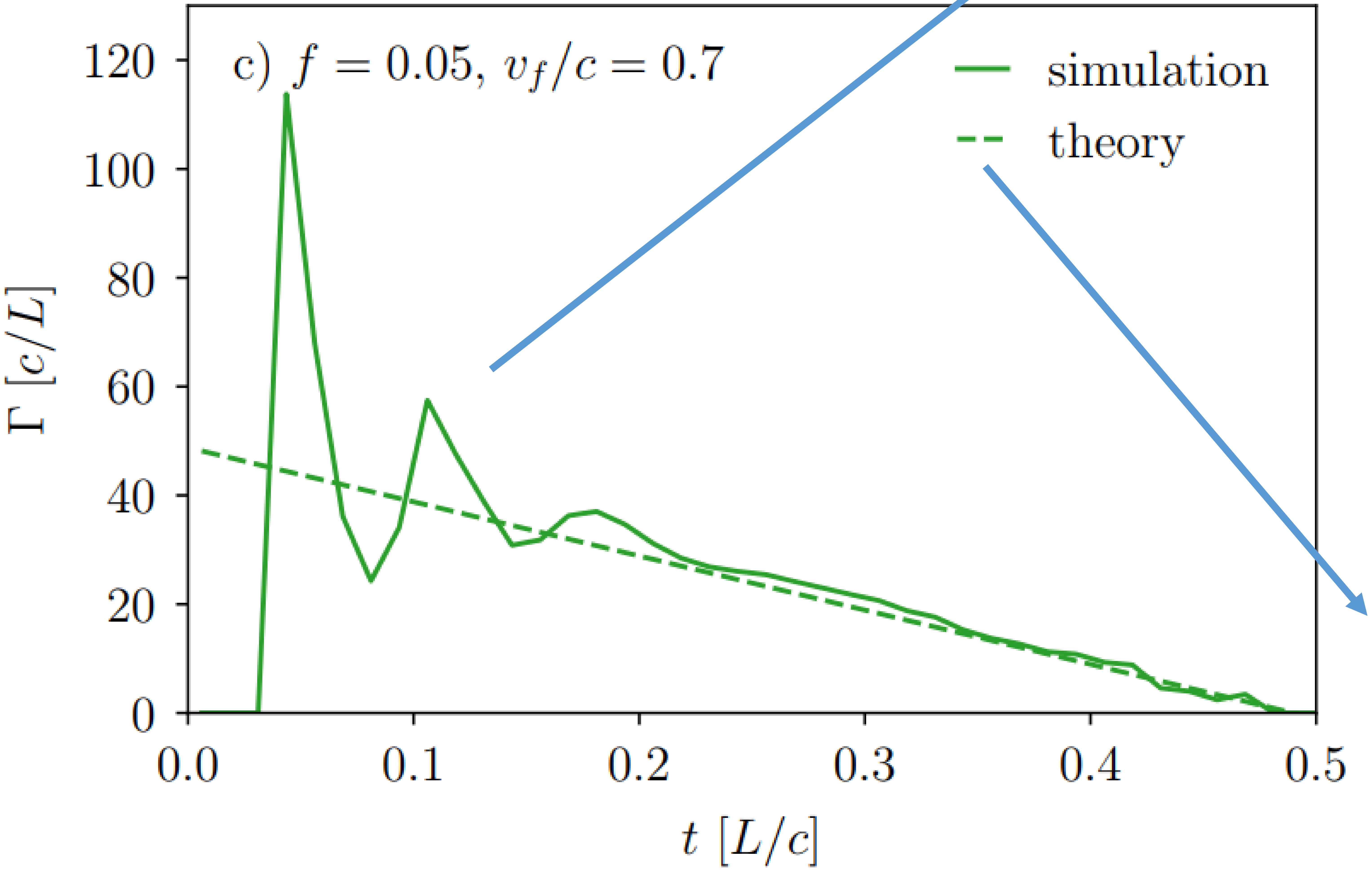
$$\Gamma(t)t_a(t) \simeq W(1/f)$$



b) $f = 0.05$
10^7
10^5
10^3
10^1
N_-
0.0
0.1
0.2
0.3
0.4
0.5
$t\ [L/c]$
$v_f/c = 0.5$
$v_f/c = 0.6$
$v_f/c = 0.7$
$v_f/c = 0.8$
$v_f/c = 0.9$

t ↑ Growing ↓

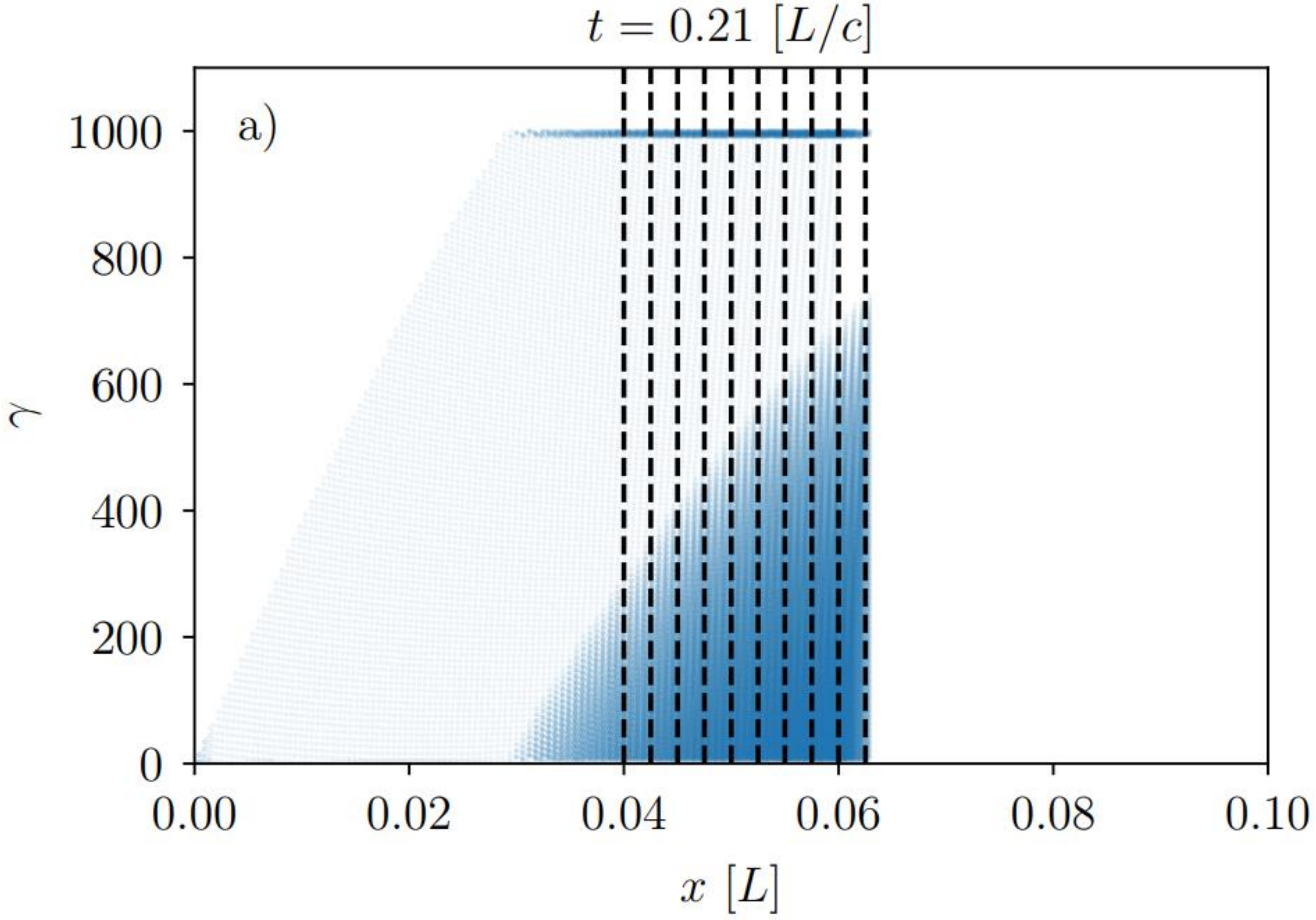
Not yet purely exponential.



$$\Gamma(t)t_a(t) = \frac{1}{1+\psi(t)} W\left(\frac{1+\psi(t)}{f}\right)$$

Simulation (2) —— an **initially uniform positron distribution** in linear E field.

Positrons flow towards x~0 ➔ Electrons produced ➔ Electrons produce **layers** of cascade.

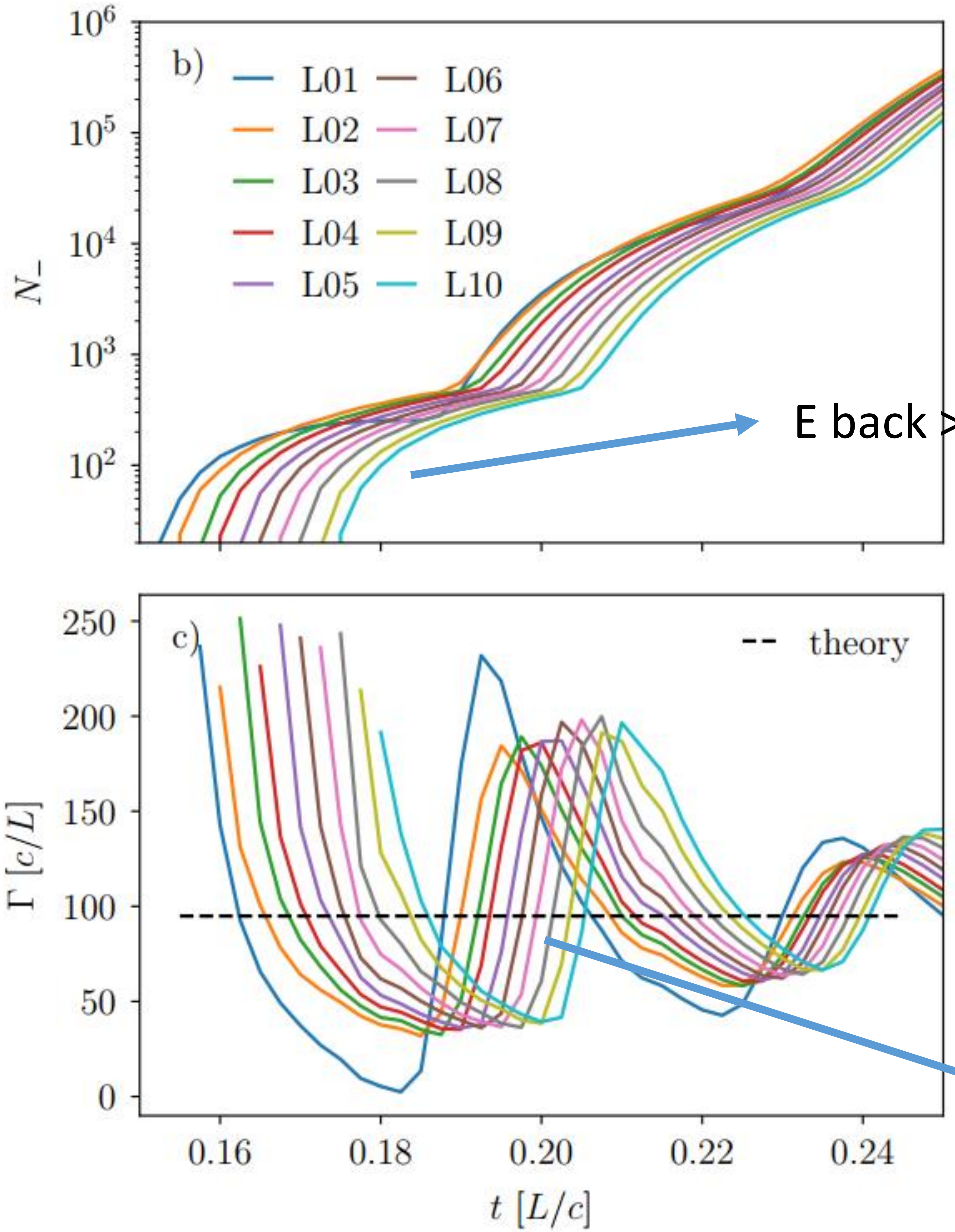


… 5 4 3 2 1

$f=0.01$

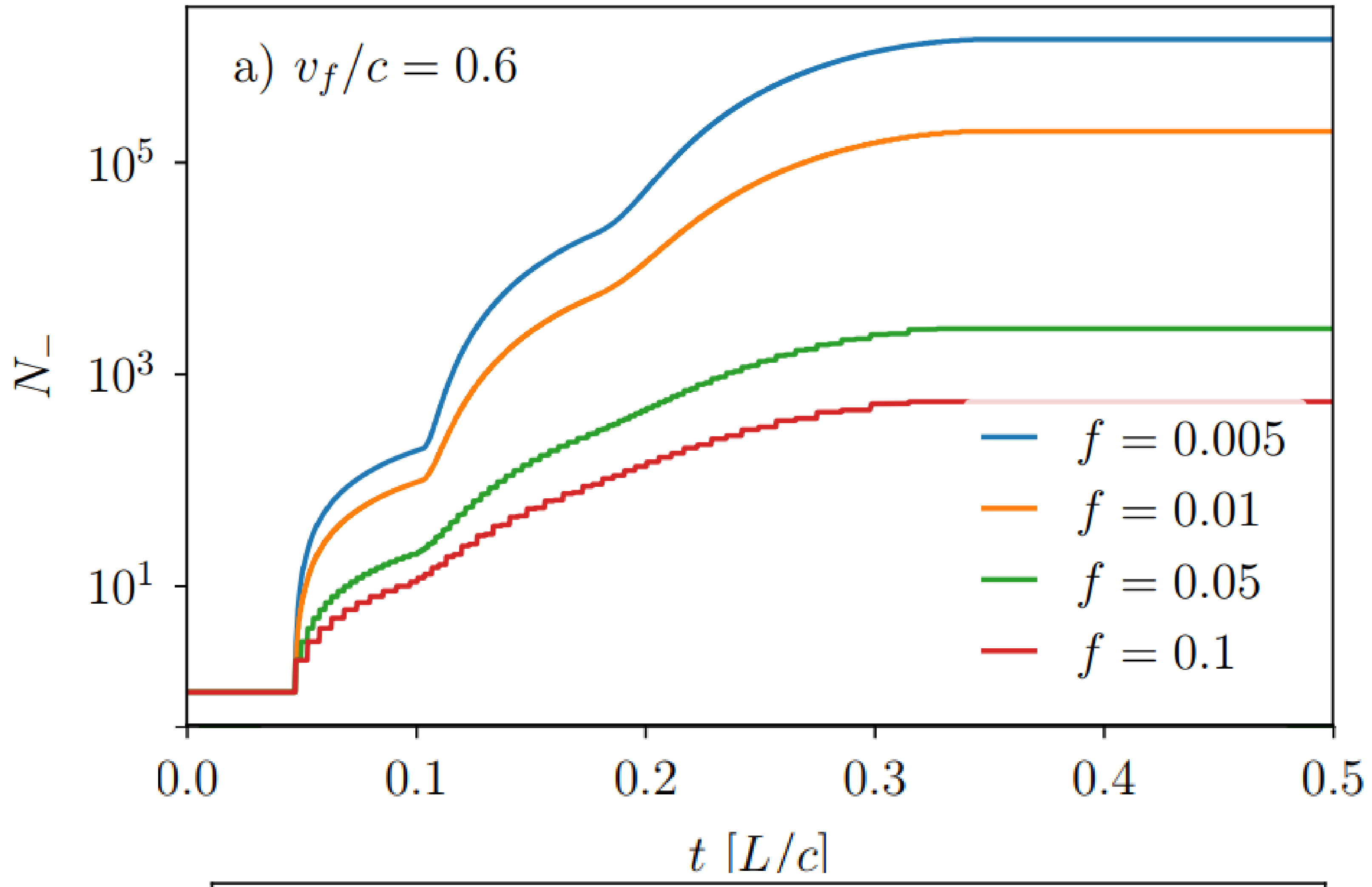
$v_f/c=0.9$

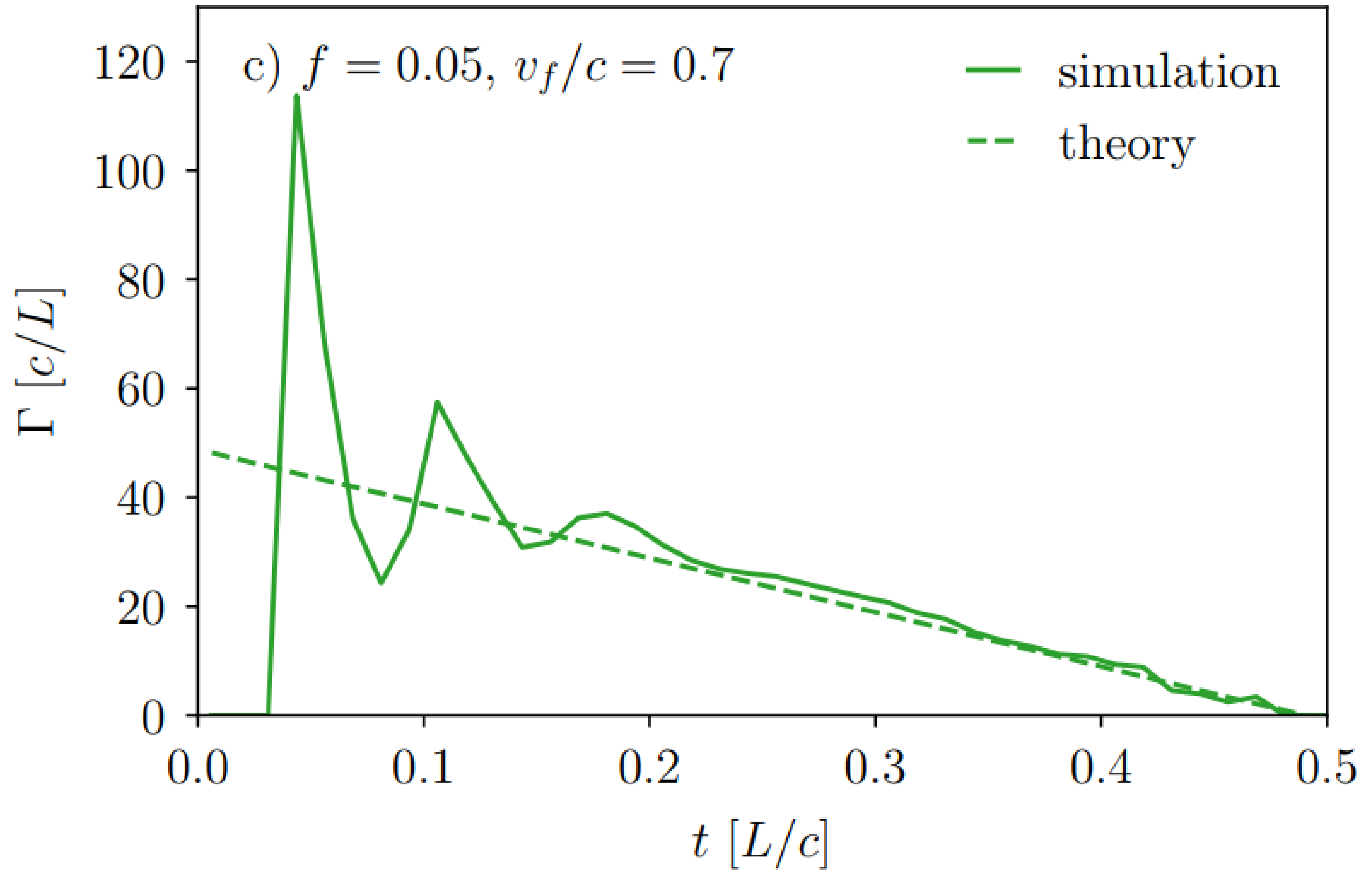
$\Delta x_L=0.0025L$



E back > E front

Each layer:

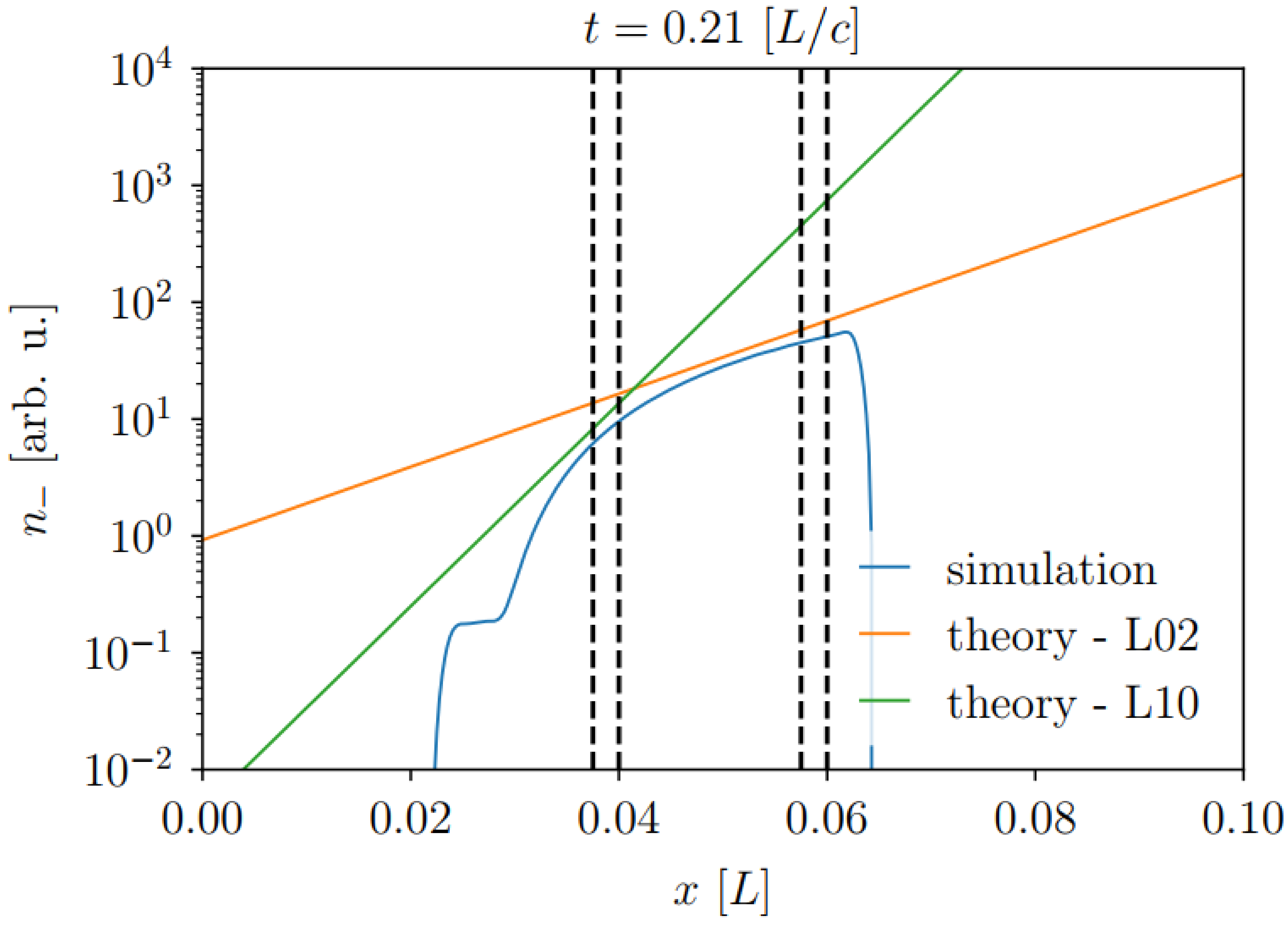




$$\Gamma(t) \simeq \frac{W(1/f)}{t_a^*}$$

Varying growth rate ➔ non-uniform electron density distribution.

Electron density spatial profile:



Electrons farther from the front are created with larger time lags.

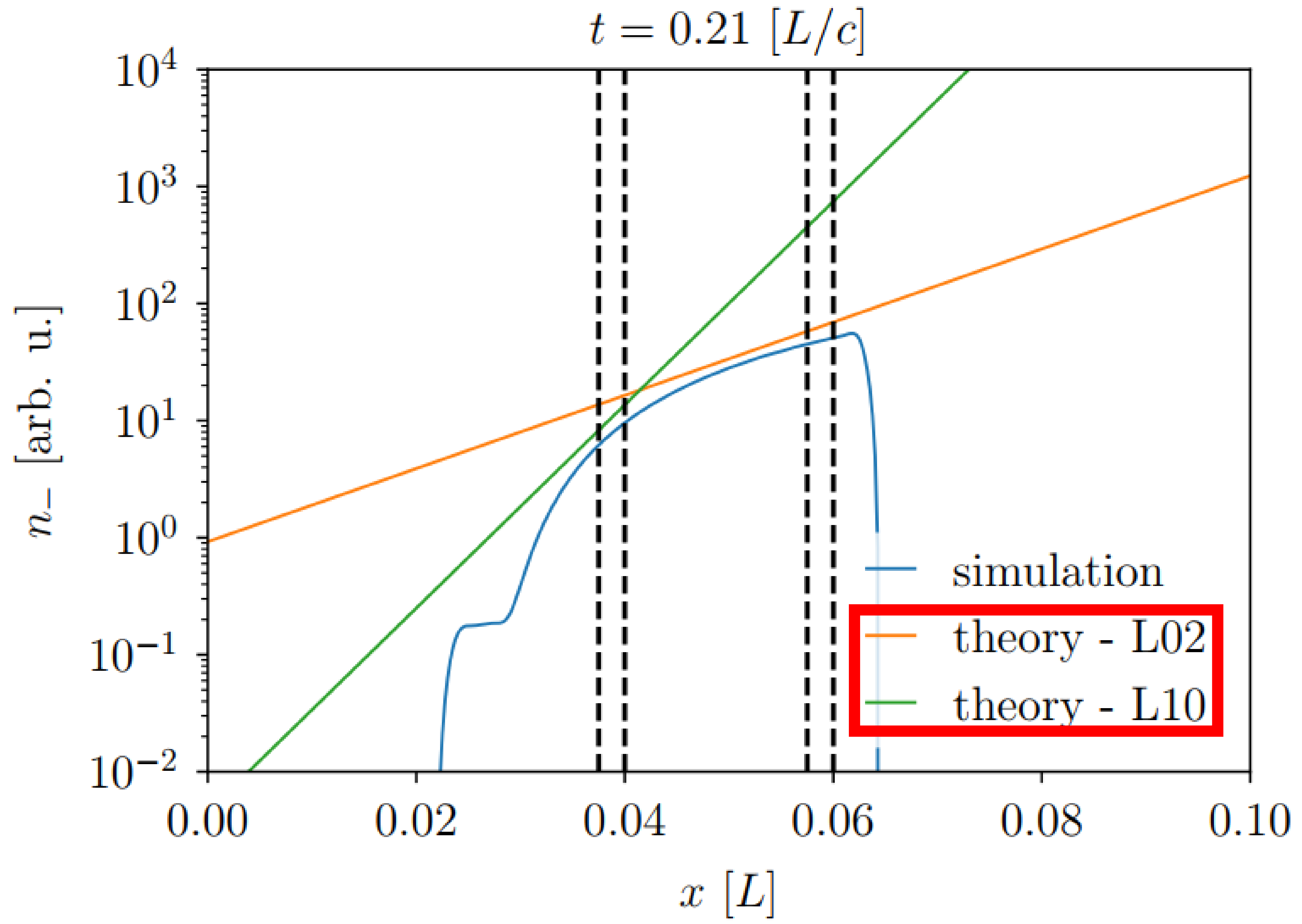
Consider time lag ≈ $\Delta x_L/c$ between layers, and we have:

$$n_{-,k}(t) \simeq n_{-,k+1}(t) \exp\left(\Gamma(t)\Delta x_{\mathrm{L}}/c\right)$$

$\Delta x_L \rightarrow 0$:

$$n_{-}(x,t) \propto \exp(\Gamma(t)(t + x/c))$$

$$t\dot{\Gamma}(t)/\Gamma(t) \ll 1$$



To calculate the
E field screening time.

At any position, when $n_{-}(-e)c = j_m$, the E field there get screened.

$$\frac{\partial E}{\partial t} = -4\pi(j - j_{\rm m})$$

We have: $j_m \sim \rho_{GJ} c$

Former simulation (Timokhin 2010; Timokhin and Aron 2012; Cruz, Grismayer and Silva 2021) shows:

$$n_{-0} \simeq 0.01 \sim 0.1 \frac{|\rho_{GJ}|}{e}$$

The screen time is about:

$$t_s \simeq \frac{1}{\Gamma} \sim \frac{t_a^*}{W(1/f)} \sim t_a^* \simeq 10^{-9} \sim 10^{-6} s$$

Consistent with (Timokhin and Harding 2015).

IV. Conclusion:

Such heuristic model provides an important way to associate QED processes with plasma kinetic effects.

Some more complex settings may be applied in the future.